WU-AP/53/95
astro-ph/9511143

An Inflationary Model with an Exact Perturbation Spectrum

Richard Easther [1]

Department of Physics,
Waseda University, 3-4-1 Okubo, Shinjuku-ku,
Tokyo, Japan.

Abstract

We present a new, exact scalar field cosmology for which the spectrum of scalar (density) perturbations can be calculated exactly. We use this exact result to the probe the accuracy of approximate calculations of the perturbation spectrum.

PACS: 04.20.Jb 98.80.C

[1] easther@cfi.waseda.ac.jp

1 Introduction

The inflationary paradigm was originally motivated by its ability to solve the "initial conditions" problems associated with the standard model of the Big Bang [1]. However it was quickly realised that an inflationary epoch would also produce primordial density perturbations and may be able to explain both the observed clustering of galaxies and the (then unobserved) anisotropies in the Cosmic Microwave Background (CMB).

Insisting that inflation produces the observed spectrum of primordial perturbations is a more demanding requirement than merely providing the approximately 60 e-foldings of inflation needed to solve the various initial conditions problems. Consequently, the focus of much present work is on the density perturbation spectra produced by different inflationary models. This is particularly true of slow-rolling inflation, in which the scalar field evolves continuously. The consistency of slow-rolling inflation can be directly tested through CMB observations [2], and in principle the potential can be reconstructed [3, 4], opening a window into a GUT-scale particle physics.

In order to do this, accurate calculations of the perturbation spectra produced during inflation are required. Stewart and Lyth [5] give a second order calculation of the perturbation spectra for a general potential. Exact scalar field cosmologies have been widely studied, in for instance, [6, 7, 8, 9, 10, 11, 12], but power-law inflation [13, 14, 15] remains the only only inflationary model for which the perturbation spectrum has been obtained exactly [16, 17].

The purpose of this paper is to present a new scalar field cosmology for which the spectrum of scalar perturbations can be obtained analytically. This solution therefore provides a second example of a slow-rolling inflationary cosmology with an exact perturbation spectrum. While the perturbations produced are not within the parameter range permitted by observation, this model extends our understanding of the problem and can be used to probe the validity of the approximation schemes used to tackle the more general problem.

2 The Equations of Motion

For a scalar field, ϕ with potential $V(\phi)$ in a spatially flat Robertson-Walker metric we have

$$H^2 = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right), \tag{1}$$

$$\frac{\ddot{a}}{a} = \frac{1}{3}\left(V(\phi) - \dot{\phi}^2\right), \tag{2}$$

where a is the scale factor and $H = d\ln a/dt$, is the Hubble parameter. From these equations we obtain the equation of motion for the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \tag{3}$$

As is often the case when dealing with exact scalar field cosmologies it will be useful to parametrise the motion in terms of the field, ϕ [7, 10, 15, 18]. From equations (1) and (2) we deduce that $d\phi/dt = -2dH/d\phi$, leading to

$$\begin{aligned} V(\phi) &= 3H^2 - 2H'^2, & (4)\\ \frac{a}{a_0} &= \exp\left(-\frac{1}{2}\int_{\phi_0}^{\phi}\frac{H}{H'}d\phi\right), & (5)\\ t &= -\frac{1}{2}\int_{\phi_0}^{\phi}\frac{1}{H'}d\phi, & (6) \end{aligned}$$

where a dash denotes differentiation with respect to ϕ. If we specify $H(\phi)$ we can immediately obtain the corresponding potential and evolution. The equation governing the evolution of scalar perturbations with wavenumber k is [5, 19, 20, 21]

$$\frac{d^2u_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2z}{d\eta^2}\right)u_k = 0 \tag{7}$$

where $\eta = \int 1/adt$ is the conformal time and $z = a\dot{\phi}/H$. Furthermore, we have the boundary conditions

$$\begin{aligned} u_k \to \frac{1}{\sqrt{2k}}e^{-ik\eta} \quad &, \quad aH \ll k, & (8)\\ u_k \propto z \quad &, \quad aH \gg k, & (9) \end{aligned}$$

which guarantees that the perturbation behaves like a free field well inside the horizon and is fixed at superhorizon scales.

In practice, however, we are interested in the spectrum P_s and index, n_s which are given by

$$\begin{aligned} P_s^{1/2}(k) &= \sqrt{\frac{k^3}{2\pi^2}}\left|\frac{u_k}{z}\right|\Bigg|_{aH=k}, & (10)\\ n_s &= 1 + \frac{d\ln P_s}{d\ln k}. & (11) \end{aligned}$$

The form of equation (7) can be simplified by defining $U_k = u_k/z$, and

$$\begin{aligned} &\frac{d^2U_k}{d\eta^2} + \frac{2}{z}\frac{dz}{d\eta}\frac{dU_k}{d\eta} + k^2U_k = 0\\ \Rightarrow\quad &\frac{d^2U_k}{d\eta^2} + 4Ha\left[\frac{1}{2} + \left(\frac{H'}{H}\right)^2 - \frac{H''}{H}\right]\frac{dU_k}{d\eta} + k^2U_k = 0. \qquad (12) \end{aligned}$$

We now turn our attention to the particular case where z is a constant, which is equivalent to demanding that the term in square brackets in equation (12) vanishes.

3 The Inflationary Model

In order to construct a model with an exact perturbation spectrum we demand that z is constant. This is equivalent to requiring that H satisfy the differential equation

$$\frac{1}{2} + \left(\frac{H'}{H}\right)^2 - \frac{H''}{H} = 0 \tag{13}$$

which has the solution

$$H(\phi) = B \exp\left(\frac{\phi^2}{4} + A\phi\right). \tag{14}$$

The values of the integration constants A and B are arbitrary, but we can set $A = 0$ without loss of generality, as it can be recovered by making a linear shift of the field. From equations (4) to (6) we derive the corresponding exact scalar field cosmology,

$$\begin{aligned} V(\phi) &= B^2\left(3 - \frac{\phi^2}{2}\right)\exp\left(\frac{\phi^2}{2}\right), &(15)\\ a(\phi) &= \frac{\phi_0}{\phi}, &(16)\\ t(\phi) &= \frac{1}{2B}\left[\mathrm{Ei}\left(-\frac{\phi_0^2}{4}\right) - \mathrm{Ei}\left(-\frac{\phi^2}{4}\right)\right], &(17)\end{aligned}$$

where Ei is the exponential integral function. The conformal time is

$$\eta(\phi) = \frac{B\sqrt{\pi}}{\phi_0}\left[\mathrm{erf}\left(-\frac{\phi_0}{2}\right) - \mathrm{erf}\left(-\frac{\phi}{2}\right)\right]. \tag{18}$$

At late times, or as ϕ goes to zero, the conformal time tends to a constant value.

The cosmological properties of this solution are quickly derived. The potential, shown in Figure 1, is not bounded below. However, for this solution the total energy density is always positive as the kinetic energy is very large in the region where $|\phi|$ is large. The motion is not inflationary at all times. By definition, inflation occurs when $\ddot{a} > 0$, or when $\epsilon < 1$, where

$$\epsilon = 2\left(\frac{H'}{H}\right)^2. \tag{19}$$

Thus inflation occurs only when $|\phi| < \sqrt{2}$. If this model was to produce *all* the 60 e-foldings of inflation needed to solve the initial conditions problems in the standard model of cosmology, ϕ must evolve to be unreasonably close to zero in view of the typical size of the perturbations in the field.

This exact inflationary model is similar to one previously discussed by Barrow [12], and can clearly be generalised in a number of ways. In particular, taking the Hubble parameter to be of the form $H(\phi) = B\exp(\phi^2/m)$ gives a similar potential. However, in this paper we will focus on the case where z is constant, which requires H to have the form of equation (14).

4 The Perturbation Spectrum

We now turn our attention to the spectrum of scalar perturbations produced by this model. The solution to equation (12) is simple, as the first derivative term drops out and we find

$$u_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{20}$$

for the growing mode, after we have imposed the boundary conditions. In terms of the conformal time, this solution has the special property that the perturbations always evolve according to the equation of motion for a free field. However, since η tends to a constant value u_k is essentially fixed at super-horizon scales, as we would expect.

We can immediately calculate the spectral index for these perturbations, giving

$$n_s = 3. \tag{21}$$

Note that this value is both exact and independent of ϕ. A flat, or scale-free, spectrum has an index of unity. Hence this spectrum is "blue" as it possesses more power at large values of k, or small scales. Astrophysical constraints, both from the CMB data [22] and bounds on the density of small primordial black holes [23] rule out such a large value of n_s in the physical universe. The spectrum of the tensor, or gravitational, perturbations has not been obtained exactly. However when $\phi \approx 0$, the expansion is roughly exponential and we therefore expect the tensor perturbation spectrum to be flat.

This model can be used to probe the accuracy of the first and second order expressions for the scalar perturbation spectra. Since this exact result pertains to a particular potential it does not provide a definitive test, but the results are reassuring. Written in terms of derivatives of $H(\phi)$ the perturbation spectrum is [5]

$$n_{s,1}(\phi) = 1 - 8\left(\frac{H'}{H}\right)^2 - 4\frac{H''}{H}, \tag{22}$$

$$n_{s,2}(\phi) = n_{s,1} - 8(1+c)\left(\frac{H'}{H}\right)^4 - 2(3-5c)\frac{H'^2 H''}{H^3} + 2(3-c)\frac{H''' H'}{H^2}. \tag{23}$$

The subscripts 1 and 2 denote the first and second order values, respectively, and $c = 4(\ln 2 + \gamma) - 5 \approx 0.081$, where γ is the Euler-Mascheroni constant.

In Figure 2 the first and second order results are plotted for the values of ϕ during which this solution is inflationary. We can immediately see that the second order result gives a much more accurate estimate of the index than the first order value, remaining within approximately 10% of the exact value when $\phi \leq 1$. The departures from slow roll for this potential are large, even though the expansion of the scale factor is approximately exponential.

5 Discussion

The exact scalar field cosmology developed in this paper adds another specimen to the collection of analytic inflationary models that have been discovered over the last decade. Unusually, however, the mode equation for the scalar perturbations can also be solved, making this only the second model (after power-law inflation) for which the exact value of the spectral index of the scalar perturbations is known. The spectrum has an index of 3, which is significantly different from the scale free value of unity, and is too large to satisfy the observational bounds on the primordial spectrum. However, this solution is useful as it can be used to test approximate calculations of the spectral indices for slow-rolling inflation. Since the slow-rolling conditions are badly violated by this model the test is a comparatively stringent one, and the first order expression does not give good agreement with the exact result. However, the second order expression for the scalar index is able to match the exact result to within 10% over most of the inflationary epoch, a result which will add confidence to its use in more realistic situations.

Acknowledgements

The author thanks Ed Copeland for a valuable discussion and is supported by a JSPS post-doctoral fellowship and the Grant-in-Aid for JSPS fellows (0694194).

References

[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[2] S. Dodelson, L. Knox, and E. W. Kolb, Phys. Rev. Lett. **72**, 3444 (1994).

[3] E. J. Copeland, E. W. Kolb, A. R. Liddle, and J. E. Lidsey, Phys. Rev. D **48**, 2529 (1993).

[4] E. J. Copeland, E. W. Kolb, A. R. Liddle, and J. E. Lidsey, Phys. Rev. D **49**, 1840 (1994).

[5] E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).

[6] M. Madsen, Gen. Rel. Grav **18**, 879 (1986).

[7] A. G. Muslimov, Class. Quantum Grav. **7**, 231 (1990).

[8] G. F. R. Ellis and M. S. Madsen, Class. Quantum Grav. **8**, 667 (1991).

[9] J. E. Lidsey, Class. Quantum Grav. **8**, 923 (1991).

[10] R. Easther, Class. Quantum Grav. **10**, 2203 (1993).

[11] J. D. Barrow, Phys. Rev. D **48**, 1585 (1993).

[12] J. D. Barrow, Phys. Rev. D **49**, 3055 (1994).

[13] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[14] F. Lucchin and S. Matarrese, Phys.Lett. B **164**, 282 (1985).

[15] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990).

[16] L. F. Abbott and M. B. Wise, Nuc. Phys. B **244**, 541 (1984).

[17] D. H. Lyth and E. D. Stewart, Phys. Lett. B **274**, 168 (1992).

[18] J. Lidsey, Phys. Lett. B **273**, 42 (1991).

[19] V. F. Mukhanov, Phys. Lett. B **218**, 17 (1989).

[20] N. Makino and M. Sasaki, Prog. Theor. Phys. **86**, 103 (1991).

[21] V. Mukhanov, H. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[22] C. L. Bennett *et al.*, Ap. J (to appear).

[23] B. J. Carr, J. H. Gilbert, and J. E. Lidsey, Phys. Rev. D **50**, 4853 (1994).

Figure Captions

Figure 1 The potential, equation (15), is plotted for $B = 1$. The solution is only inflationary when ϕ is near the origin.

Figure 2 The first order (short dashes) and second order (long dashes) results for the perturbation spectrum are plotted, with the exact value of 3 shown for reference, against the values of ϕ for which the exact solution is inflationary.

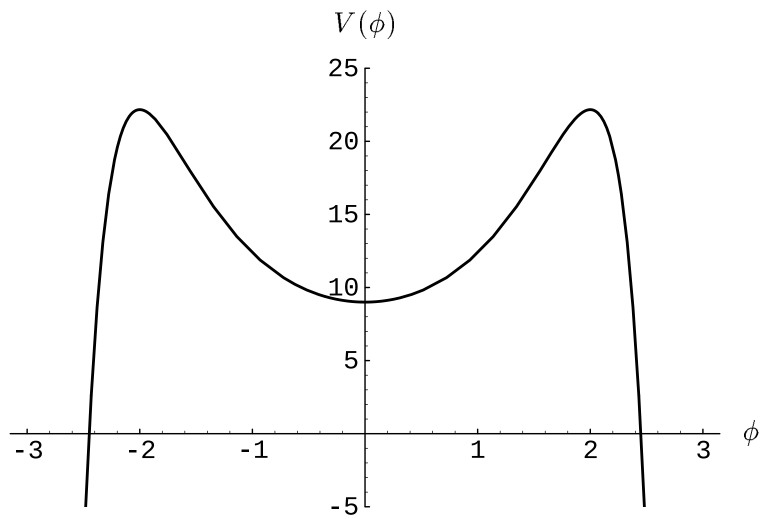

Figure 1

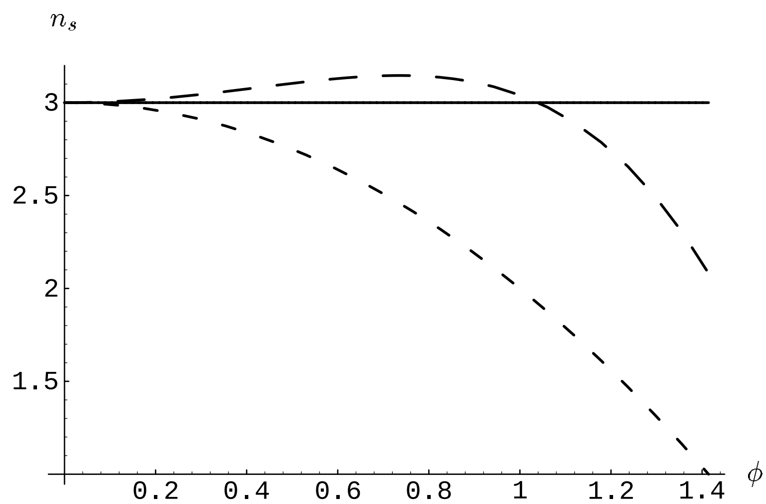

Figure 2